Mathematics of Music

Statement of Rationale

Overview:

In this interdisciplinary course, we will explore some of the connections between mathematics and music. We will do this on different levels. First, we will see how fundamental concepts in music (for example, rhythm, time signature, scales, keys, intervals, intonation, tuning and symmetry) are based on mathematical principles (geometric series, least common multiples, sine functions, rational numbers, irrational numbers and group theory, to name but a few). On a physical level, we will study the science of sound and the mathematics underlying sound waves and pitch. Lastly, we will investigate how some composers have based their creations on mathematical concepts. Examples include Schoenberg and his twelve-tone music and modern composer Xenakis, who has used computers and probability theory to create "stochastic music".

Class meetings will consist of lecture time and group activities. We will also spend time listening to sound samples illustrating concepts and techniques discussed in lecture time and group activities.

Apart from studying these connections in class, students will also get to explore them in practice. As part of the course, students will attend at least two local music productions or recitals and report on the mathematical connections they observed. They will also get to incorporate some of these mathematical connections in their own composition, which will serve as their final project in this course.

Course Learning Outcomes (CLO's):

Formally, the course objectives described above can be summarized in the following course learning outcomes. More details about the readings, assessment activities and specific topics to be covered are given below as well. By the end of this course, students should be able to:

	CLO	Instructional Activity	Assessment
(i)	demonstrate an understanding of the multiple connections between mathematics and music;	lecture group activities listening activities reading	class participation homework assignments tests final project
(ii)	develop an understanding of music theory and a deeper appreciation for music;	lecture group activities listening activities reading concert attendances	class participation homework assignments tests final project concert reports
(iii)	develop skills in analytical thinking, critical thinking and abstract reasoning;	lecture group activities reading concert attendances	class participation homework assignments tests concert reports
(iv)	integrate their artistic and analytical skills;	group activities listening activities	class participation homework assignments final project
(v)	reflect upon the Triune God, the Creator of music and mathematics, and their identities as followers of Christ.	lecture group activities	class participation

Required resources:

1. *From Music to Mathematics: Exploring the Connections*, by Gareth E. Roberts.
2. *Music and Mathematics: From Pythagoras to Fractals*, edited by John Fauvel, Raymond Flood and Robin Wilson.
3. *Music: A Mathematical Offering*, by Dave Benson.
   Published online by the author at https://homepages.abdn.ac.uk/d.j.benson/pages/html/maths-music.html.
4. Staff paper.

Assessment:

Students will be assessed as follows:

- Class participation (10% of grade):
  Active participation is essential for the success of this course. Participation may take a variety of forms: asking a clarifying question, offering examples, critiquing an idea, actively engaging other students in a small group, thoughtfully answering written prompts, or otherwise actively engaging material and contributing to the discussion. Participation will be graded daily on a 5-point scale.
- Homework assignments (20% of grade):
  Students will complete written homework assignments on a regular basis (7-8 assignments during

the course of the semester). The assignments will require students to apply theoretical concepts (from mathematics and music theory) in specific contexts, and to provide arguments and explanations about abstract concepts in mathematics, music theory and their intersection. A sample homework assignment is attached.

- Tests (2 × 20% of grade):
  Students will take two tests during the course of the semester, which will test their understanding of the music and mathematics concepts discussed in class, as well as the connections between the two subjects.

- Concert reports (2 × 5% of grade):
  Students will be required to attend two musical performances during the semester and turn in a typed 1-2 page report. The purpose of these reports is to observe connections to the course material, to enhance their musical appreciation, and to support their fellow students and the arts.

- Final project (20% of grade):
  As a conclusion of the course, students will complete a final project consisting of a musical composition and performance demonstrating some of the mathematical concepts they have learned in the course. They will also write a short report to explain the mathematical connections and rationale in their work. Performances of the final project will take place during the scheduled final exam period.

Tentative schedule:

Week 1	Introduction
	Rhythm (Roberts Chapter 1)
Week 2	Rhythm (Roberts Chapter 1)
	Basic Music Theory (Roberts Chapter 2)
Week 3	Basic Music Theory (Roberts Chapter 2)
Week 4	The Science of Sound (Roberts Chapter 3, Benson Chapters 1, 3)
Week 5	The Science of Sound (Chapter 3, Benson Chapters 1, 3)
	Test 1
Week 6	Tuning and Temperament (Roberts Chapter 4)
Week 7	Tuning and Temperament (Roberts Chapter 4)
Week 8	Musical Group Theory (Roberts Chapter 5)
Week 9	Musical Group Theory (Roberts Chapter 5)
Week 10	Change (Bell) Ringing (Roberts Chapter 6, Fauvel Chapter 7)
Week 11	Test 2
	Twelve-Tone Music and Serialism (Roberts Chapter 7)
Week 12	Twelve-Tone Music and Serialism (Roberts Chapter 7)
	Mathematical Modern Music (Roberts Chapter 8, Fauvel Chapter 8)
Week 13	Mathematical Modern Music (Roberts Chapter 8, Fauvel Chapter 8)
Week 14	Final Project: Mathematical Composition
Week 15	Final Project: Mathematical Composition

General Education: Quantitative and Analytical Reasoning Criteria:

For a course to fulfill the GE requirement *Quantitative and Analytical Reasoning*, students should be able to:

- make use of mathematical (including statistical) models for physical or social systems -and/or-compute and interpret numeric data, summative statistics and/or graphical representations:
  In this course, students will study the mathematical models describing sound waves, how we hear sound and the musical concepts of time signatures, pitch and rhythm. This is reflected in CLO (i) in the table above with specifc instructional activities and assessments.

- reflect on the strengths and weaknesses of particular quantitative models or methods as tools in the natural and social sciences:
  In this course, students will compare the use of different mathematical models in tuning instruments (including just intonation and equal temperament) and in music composition (including serialism and stochastic processes). This is reflected in CLO (i) and (iii) in the table above.

- be able to interpret, reflect on, and use quantitative models and data in public, vocational, and/or private decision making:
  Throughout the course, but specifically in their final project, students will be required to make use of different mathematical models in creating their own composition, and to interpret and explain (both written and verbally) their decisions in using specific models in their composition. This is reflected in CLO (i), (iii) and (iv) in the table above.

General Education: Quantitative and Analytical Reasoning Student Learning Outcome:

In a course that satisfies the GE requirement *Quantitative and Analytical Reasoning*, students will apply relevant scientific, mathematical and logical methods to analyze and solve problems effectively and be able to utilize the results appropriately when making decisions. In this course, specifcally, students will apply relevant mathematical and logical methods to analyze sound and concepts in music theory and composition effectively and be able to utilize the results appropriately when making decisions in this field (for example, when creating their own composition). This is reflected in CLO (i), (iii) and (iv) in the table above.

General Education: Reasoning Abstractly Criteria:

For a course to fulfill the GE requirement *Reasoning Abstractly*, students should be able to:

- identify instances of abstract reasoning about abstract objects or concepts (in the form of arguments, explanations, proofs, analyses, modeling, or processes of problem solving) and can distinguish premises from conclusions (or their analogues):
  In this course, students will learn to identify and understand mathematical arguments, proofs and models that analyze mathematical concepts in musical structures (including rhythm, time signature and pitch). This is reflected in CLO (i) and (iii) in the table above.

- construct an instance of valid reasoning about abstract objects or concepts (in the form of arguments, explanations, proofs, analyses, modeling, or processes of problem solving):
  Students will use mathematical arguments and explanations to analyze mathematical concepts in musical structures (including rhythm, time signature and pitch). This is reflected in CLO (i) and (iii) in the table above.

- distinguish valid forms of reasoning about abstract objects or concepts (in the form of arguments, explanations, proofs, analyses, modeling, or processes of problem solving) from invalid and/or fallacious forms of reasoning:

In engaging with different mathematical arguments, explanations, models and proofs, students will learn to distinguish valid forms of reasoning about abstract mathematical concepts in music theory from invalid ones. This is reflected in CLO (i) and (iii) in the table above.

General Education: Reasoning Abstractly Student Learning Outcome:

In a course that satisfies the GE requirement *Reasoning Abstractly*, students will be able to recognize, construct, and evaluate instances of abstract reasoning. In this course, specifcally, students will learn to recognize, construct and evaluate instances of abstract mathematical reasoning in the context of music theory and composition. This is reflected in CLO (i) and (iii) in the table above.

Written Homework Assignment 1

How to do this assignment:

- Complete the problems below. Submit a hard copy of your solutions (either handwritten or typed up) at the start of class.
- Show all work for full credit.
- You are welcome to get help from me or collaborate with your class mates on this assignment, as long as you write and turn in your own set of solutions and list the names of your collaborators on your solutions.
- Late homework will not be accepted without warning – if you cannot make it to class to submit your homework, please contact me beforehand so that we can make alternative arrangements.

Why I want you to do this assignment:

- You will apply the definitions of geometric sequences and series to specific contexts.
- You will apply the definitions of musical notation, time signatures and beats to specific contexts.
- You will gain experience constructing abstract arguments.

How I will grade this assignment:

- Each problem is worth 2 points.
- For each problem, I will award 2 points if your work is perfect, 1.5 points if you have a couple of minor errors, and 1 point if you have major errors. No points will be awarded for answers without justification or incomplete work.

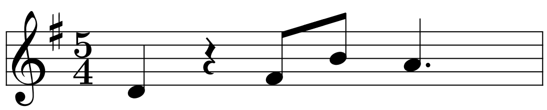
1. Write out the first six terms of a geometric sequence with common ratio $r = 3$ and whose first term is $a_0 = \frac{2}{9}$. Check that each term, other than the first or the last, is the geometric mean of its adjacent neighbors.
2. Any three consecutive terms in a geometric sequence with common ratio r can be written as a, ar, ar^2. Compute the geometric mean of a and ar^2 and check that it is equal to the middle term ar.
3. Suppose that the length of a quarter note corresponds to one beat. How many beats does a triple-dotted quarter note get? How many beats does a double-dotted half note get?
4. Suppose that the length of a quarter note corresponds to one beat. How many beats does a double-dotted eighth rest get? How many beats does a double-dotted sixteenth rest get?
5. Measures 356-357 of the vocal parts (soprano, alto, tenor and bass) of the *Dies irae* from Verdi's *Requiem* are shown below. How many beats should the sopranos, altos and tenors hold their opening notes in measure 356?

6. Write out the first eight terms of the infinite geometric series with $a_0 = 250$ and $r = -\frac{1}{5}$. What is the sum of the infinite series?

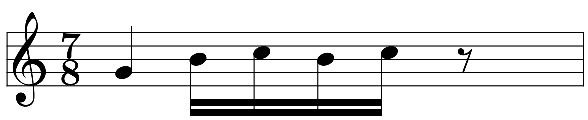
7. Find the sum of the infinite geometric series $9 - 6 + 4 - \frac{8}{3} + \frac{16}{9} - + \cdots$.

8. How many quarter notes are required to fill up a measure in $\frac{7}{2}$ time? How many eighth notes are needed to fill up the same measure?

9. The following measures of music are incomplete. In each case, the last note of the measure has been removed. Give the notation and name of the note (e.g., quarter note, dotted eighth note, etc.) that completes the measure.

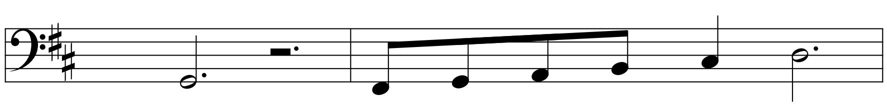


(a)

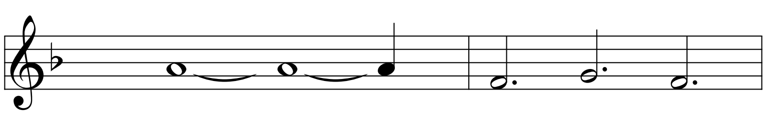


(b)

10. In each of the following excerpts of music, the time signature has been omitted. Assuming that a quarter note equals one beat, provide the correct time signature at the start of each excerpt.



(a)



(b)